Statistics
Homework 3: Discrete Distributions

Javier Tasso

1. Solve the following exercises of chapter 3 of Devore, Berk, Carlton's *Modern Mathematical Statistics with Applications* (third edition): 2, 6, 7, 11, 12, 13, 14, 18, 21, 23, 28, 29, 30, 31, 32, 45, 63, 65, 67, 81, 89, 91, 93, 99, 108, 109, 110, 112, 120.

2. Use properties of the variance to prove that $V(X) = V(-X)$.

3. Consider a binomial distribution $X \sim \text{Bi}(n, p)$. It's well known that $V(X) = np(1-p)$. For what value of p does the variance attain its maximum? For what values of p does the variance attain its minimum?

4. Consider $Y = X_1 + X_2 + \cdots + X_n$ where each $X_i \sim \text{Ber}(p)$ independent from each other. Calculate $E(X)$ and $V(X)$ using properties of the expected value and the variance.

5. Consider a Geometric random variable $X = 0, 1, 2, 3, \ldots$ with $P(X = k) = (1-p)^k p$. Using the law of iterated expectations, calculate

   (a) Its expected value.

   (b) Its variance.

   Hint: Condition on getting the success in the first trial $X = 0$ or later $X > 0$. Conditional on $X = 0$ the variable is no longer random. Conditional on $X > 0$ the random variable becomes $1 + X$.

Answers

1. You can find answers to the odd numbered problems at the end of the textbook.

2. The gender of a randomly selected student ($X = 1$ if female, $X = 0$ otherwise). The flip of a coin ($X = 1$ for heads). Whether an item passes a quality control ($X = 1$ if it does).

6. $X = 1, 2, 3, \ldots$. One outcome could be RRL with $X = 3$. Another outcome could be ARL with also $X = 3$.

12. a. 0.83
    b. 0.17
    c. 0.66 and 0.27

14. a. $k = 1/15$
    b. 6/15
    c. 9/15
    d. No, because probabilities do not sum 1.

18. M can take the following values $1, 2, 3, 4, 5, 6$. Their probabilities (in that order) are $1/36, 3/36, 5/36, 7/36, 9/36, 11/36$. You are in charge of finding and graphing the cdf.

28. a. 2.06
    b. 0.9364

30. $\sigma = 2.12$, $\mu = 48.84$. We want the probability that Y is in the interval $[46.72, 50.96]$ this includes values $Y = 47, 48, 49, 50$ and the probability is 0.68.

32. $X = 0, 1$ with $P(X = 0) = 1 - p$ and $P(X = 1) = p$.
    a. $E(X^2) = 0^2 \cdot (1 - p) + 1^2 \cdot p = p$
    b. $V(X) = E(X^2) - \mu^2 = p - p^2 = p(1 - p)$
    c. $E(X^{79}) = 0^{79} \cdot (1 - p) + 1^{79} \cdot p = p$

108. a. Hypergeometric with $N = 20$, $n = 6$, and $A = 12$.
     b. 0.1192, 0.13725, 0.98184.
     c. 3.6 and 1.03007

110. a. 0.20695
     b. 0.37982
     c. At least 10 are from the second section: 0.37982. At least 10 are from the first section (or less than or equal to 5 are from the second section) 0.01399. The final probability is then: 0.39381.
     d. 9 and 1.60357
     e. 21 and 1.60357

112. Define X to be the number of patients with no adverse reaction $X = 0, 1, 2, 3, \ldots$ where 0 means the first patient already had an adverse reaction. $P(X = k) = (1 - p)^k p$. Define Y to be the number of patients treated $Y = 1, 2, 3, \ldots$ where 1 means the first patient already had an adverse reaction. $P(Y = k) = (1 - p)^{k-1} p$. These two variables are equivalent.
     a. $P(X = 4) = P(Y = 5) = 0.08192$
     b. $P(X = 4) = P(Y = 5) = 0.08192$

c. $P(X \leq 4) = P(Y \leq 5) = 0.67232$

d. $E(X) = \frac{1-p}{p} = 4$, $V(X) = \frac{(1-p)}{p^2} = 20$ and $E(Y) = \frac{1}{p} = 5$, $V(Y) = \frac{(1-p)}{p^2} = 20$.

e. Focus on Y. We expect 5. Standard deviation is $\sqrt{20} \simeq 4.5$. We want the probability that Y lies within [1,9]. This probability is: $P(Y \leq 9) - P(Y \leq 1) = 0.86578 - 0.2 = 0.66578$.

120. Let X be the spin in which I win $X = 1, 2, 3, \ldots$ with $P(X = k) = (1-p)^{k-1}p$ and $p = 1/10$.

a. $P(X = 1) = 0.1$

b. At most 5: 0.40951. Exactly 5: 0.06561. At least 5: 0.65610.

c. 10 and $\sqrt{90}$.

2. Use the formula $V(aX + b)$, where $a = -1$ and $b = 0$.

3. $p = 1/2$ for the maximum. $p = 0$ or $p = 1$ for the minimum. In the last two cases, there's nothing random about X.

4. $E(X) = np$ and $V(X) = np(1-p)$.

5. The geometric random variable doesn't have memory. Condition on getting the success in the first trial $X = 0$ vs getting it later $X > 0$.

(a) $E(X|X = 0) = 0$ (because we are conditioning on that specific value $X = 0$) and $E(X|X > 0) = E(1 + X) = 1 + E(X)$ (because starting in the next trial, the expected value is the same).

$$E(X) = p{\color{red}E(X|X = 0)} + (1-p){\color{blue}E(X|X > 0)}$$
$$E(X) = p{\color{red}0} + (1-p){\color{blue}(1 + E(X))}$$
$$E(X) = (1-p)(1 + E(X))$$

Finally solve for $E(X) = \frac{1-p}{p}$.

(b) $E(X^2|X = 0) = 0$ (because we are conditioning on that specific value $X = 0$) and $E(X^2|X > 0) = E[(1+X)^2] = 1 + 2E(X) + E(X^2) = 1 + 2\frac{1-p}{p} + E(X^2)$ (because starting in the next trial, the expected value is the same).

$$E(X^2) = p \cdot {\color{red}E(X^2|X = 0)} + (1-p) \cdot {\color{blue}E(X^2|X > 0)}$$
$$E(X^2) = p \cdot {\color{red}0} + (1-p) \cdot {\color{blue}[1 + 2\frac{1-p}{p} + E(X^2)]}$$

Solve for $E(X^2)$:

$$E(X^2) = \frac{1-p}{p} + \frac{2(1-p)^2}{p^2}$$

Use the shortcut formula:

$$
\begin{aligned}
V(X) &= E(X^2) - \mu^2 \\
&= \frac{1-p}{p} + \frac{2(1-p)^2}{p^2} - \frac{(1-p)^2}{p^2} \\
&= \frac{1-p}{p} + \frac{(1-p)^2}{p^2} \\
&= \frac{1-p}{p}\left(1 + \frac{1-p}{p}\right) \\
&= \frac{1-p}{p^2}
\end{aligned}
$$