Statistics
Homework 2: Probability

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1. Solve the following exercises of chapter 2 of Devore, Berk, Carlton's *Modern Mathematical Statistics with Applications* (third edition): 1, 3, 4, 9, 11, 13, 14, 16, 19, 23, 31, 45, 48, 49, 50, 51, 57, 69, 70, 71, 74, 75, 78, 79, 81, 117, 127

2. Prove the following statement: if $A \subset B$ then $P(A) < P(B)$.

3. It is well known that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Use this result to show:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

4. Explain intuitively why $\binom{n}{k}$ equals $\binom{n}{n-k}$.

5. Suppose $P(B) > 0$ show that $P(A|B) + P(A'|B) = 1$.

6. Prove the following statements:

   (a) If A and B are independent, then A and B' are independent too.

   (b) If A and B are independent, then A' and B' are independent too.

Answers

1. You can find answers to the odd numbered problems at the end of the textbook.

   4. a. $A = \{RRR, LLL, SSS\}$
      b. $B = \{RLS, RSL, LSR, LRS, SLR, SRL\}$
      c. $C = \{RRL, RRS, RLR, RSR, SRR, LRR\}$
      d. $D = C \cup \{LLS, LLR, LSL, LRL, RLL, SLL\} \cup \{SSL, SSR, SRS, SLS, RSS, LSS\}$
      e. $D' = A \cup B$, $C \cup D = D$, $C \cap D = C$.

   14. a. 0.65
       b. 0.35
       c. $A \cap B'$. The probability is 0.25

   16. a. 0.31
       b. 0.69
       c. 0.37

   48. a. 2520 by doing $\binom{10}{2} \cdot \binom{8}{3} \cdot \binom{5}{5}$.
       b. Simplify in the previous expression: $\frac{10! \cdot 8! \cdot 5!}{2!8! \cdot 3!5! \cdot 5!0!} = \frac{10!}{2!3!5!}$.
       c. $n!$ in the numerator counts every possible permutation of n objects, treating all arrangements as distinct. The denominator $k_1! \ldots k_r!$ corrects for over-counting due to the fact that the internal arrangement within each group does not matter.

   50. You should expect $P(A|B)$ to be higher. The universe of professional basketball players is much smaller than the universe of tall people.

   70. Let S be the event that the student is satisfied with the textbook. By the law of total probability $P(S) = 0.51$. Now update to get the posterior probabilities $P(\text{Mean}|S) = 0.3922$, $P(\text{Median}|S) = 0.2941$, $P(\text{Mode}|S) = 0.3137$. It is still more likely he used professor Mean's textbook.

   74. $P(A) = 0.5$ and $P(A|B) = 0.625$. These are different, so A and B are not independent. You can verify that $P(A) \cdot P(B)$ does not equal$P(A \cap B)$.

   78. $P(OO) = 0.44 \cdot 0.44 = 0.1936$ and $P(\text{Match}) = P(AA) + P(BB) + P(ABAB) + P(OO) = 0.3816$

2. A is contained in B. You can use a Venn diagram to verify that $B = A \cup (B \cap A')$ where A and $(B \cap A')$ are disjoint. Then:

$$
\begin{aligned}
B &= A \cup (B \cap A') \\
P(B) &= P[A \cup (B \cap A')] \\
P(B) &= P(A) + P(B \cap A')
\end{aligned}
$$

   Now notice that $P(B \cap A') > 0$. Then $P(B) > P(A)$.

3. Define $D = B \cup C$

$$
\begin{aligned}
P(A \cup D) &= P(A) + P(D) - P(A \cap D) \\
&= P(A) + P(B \cup C) - P[A \cap (B \cup C)] \\
&= P(A) + P(B) + P(C) - P(B \cap C) - P[A \cap (B \cup C)] \\
&= P(A) + P(B) + P(C) - P(B \cap C) - P[(A \cap B) \cup (A \cap C)] \\
&= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)]
\end{aligned}
$$

Rearranging you get the desired formula.

4. Suppose you have $n = 10$ people and you want to count the number of ways you can create groups of $k = 8$. By selecting groups of 8 you are also selecting groups of 2. This is way the two combinations match. If you select a group of 8 people, the remaining 2 are automatically chosen in the unselected group.

5. Recall that $P(B) = P(A \cap B) + P(A' \cap B)$. Divide both sides by $P(B)$, use the definition of the conditional probability.

6. (a) Start with $P(A) = P(A \cap B) + P(A \cap B')$, use independence, and rearrange.

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B') \\ P(A) &= P(A)P(B) + P(A \cap B') \\ P(A) - P(A)P(B) &= P(A \cap B') \\ P(A)[1 - P(B)] &= P(A \cap B') \\ P(A)P(B') &= P(A \cap B') \end{aligned}$$

The last equality shows that A and B' are independent.

(b) Focus on B' and A. From the previous property we know they are independent (because A and B were).

$$\begin{aligned} P(B') &= P(A \cap B') + P(A' \cap B') \\ P(B') &= P(A)P(B') + P(A' \cap B') \\ P(B') - P(A)P(B') &= P(A' \cap B') \\ P(B')[1 - P(A)] &= P(A' \cap B') \\ P(B')P(A') &= P(A' \cap B') \end{aligned}$$