Discrete Distributions

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Random Variables

Random Variables

A **Random Variable** X is a function that assigns a real number to each outcome of the sample space. Formally: $X : \mathcal{S} \to \mathbb{R}$.

- X is discrete if its possible values are a finite set or a countable infinite set.
  - Example: Bernoulli random variable.
- X is continuous if it takes all the values on a real interval (or union of intervals) and no possible value has positive probability.
  - Example: Uniform random variable.
- RV vs a realization of the RV.

Probability Mass Function

A probability mass function (pms) of a discrete random variable is defined by:

$$p(x) = P(X = x) = P(s \in \mathcal{S} : X(s) = x)$$

Where $p(x) \geq 0$ and $\sum_{x \in X} p(x) = 1$.

- The support of $p(x)$ consists of all values for which $p(x) > 0$.
- Example.

Moments

Cumulative Distribution Function

The cumulative distribution function (cdf) of a discrete random variable X is defined by:

$$F(x) = P(X \leq x) = \sum_{t:t \leq x} p(t)$$

- Step function.
- Non-decreasing.
- Theoretical version of the empirical cdf introduced previously.

Expected Value

Let X be a discrete random variable. Its expected value, denoted by $E(X)$ or μ is:

$$\mu = E(X) = \sum_{x \in X} x \cdot p(x)$$

- Theoretical mean.
- Describes where the distribution is centered.
- Examples.

Expected Value: Properties

Let $a, b, c \in \mathbb{R}$ and let X and Y be random variables.

- $E(c) = c$
- $E(aX + b) = aE(X) + b$
- $E(X + Y) = E(X) + E(Y)$

Expected Value: Properties II

The law of the unconscious statistician. Let $g(X)$ be a function, then:

$$E[g(X)] = \sum_{x \in X} g(x) \cdot p(x)$$

- Absolute moments $E(X)$, $E(X^2)$, $E(X^3)$,...
- Centered moments $E[(X-\mu)]$, $E[(X-\mu)^2]$, $E[(X-\mu)^3]$, ...
- And more!

Expected Value: Properties III

The law of iterated expectations. Let X and Y be random variables:

$$E(X) = E[E(X|Y)]$$

- Conditional expectation.
- Example: 6 men with heights (in cm): 175, 170, 186, 165, 182, 172 and 4 women with heights: 170, 175, 155, 160. Let X be the height of a person and $Y \in \{0, 1\}$ where $Y = 1$ mean the person is a woman.

Variance

Let X be a discrete random variable. Its variance, denoted by $V(X)$ or σ^2 is:

$$\sigma^2 = V(X) = E[(X-\mu)^2] = \sum_{x \in X} (x-\mu)^2 \cdot p(x)$$

- Second order centered moment. Centered around the expected value.
- How dispersed around μ the distribution is.
- Define the standard deviation σ as its square root.

Variance: Properties

Let $a, b, c \in \mathbb{R}$ and let X and Y be random variables.

- $V(c) = 0$
- $V(aX + b) = a^2 V(X)$
- If X and Y are independent RVs, then $V(X \pm Y) = V(X) + V(Y)$
- Shortcut formula $V(X) = E(X^2) - \mu^2$.

Some Discrete Distributions

$X \sim \text{Ber}(p)$. It can take values $X = 0, 1$ and its pmf is:

$$p(x) = \begin{cases} p^x(1-p)^{1-x} & \text{if} \quad x = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

- $X = 1$ represents a success and $X = 0$ a failure.
- $E(X) = p$ and $V(X) = p(1-p)$.
- Symmetric if $p = 0.5$.
- Variance decreases as $p \to 0$ or $p \to 1$.

Binomial Distribution

$X \sim \mathrm{Bi}(n, p)$. Repeat n identical and independent Bernoulli RVs. $X = 0, 1, \ldots, n$ with pmf:

$$p(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{if} \quad x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

- X represents the number of successes in n independent trials.
- $E(X) = np$ and $V(X) = np(1-p)$.

Poisson Distribution

$X \sim \text{Po}(\lambda)$ with $X = 0, 1, 2, \ldots$. Its pmf is:

$$p(x) = \begin{cases} e^{-\lambda} \cdot \frac{\lambda^x}{X!} & \text{if} \quad x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

- X is the number of events that occur in a fixed interval, if this events occur at a constant mean rate and are independent of the time since the previous event.
- As a limiting case of a Binomial distribution with $n \to \infty$ and $p \to 0$.
- $E(X) = V(X) = \lambda$.

Geometric Distribution

Let $X \sim \text{Geo}(p)$ be the number of trials before getting a success $X = 0, 1, 2, \ldots$ with:

$$p(x) = \begin{cases} p(1-p)^x & \text{if} \quad x = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

- $E(X) = \frac{1-p}{p}$ and $V(X) = \frac{1-p}{p^2}$.
- Show $E(X)$ using LIE.
- Why $\sum p(x) = 1$?
- Equivalent version counting the trial in which the first success happens.