Probability

Javier Tasso

University of Pennsylvania

Axioms and the Classical Definition of Probability

Sample Space and Events

- The **Sample Space** $\mathcal{S}$ of an experiment is the set of all possible outcomes of that experiment.
- An **Event** is a subset of the sample space $\mathcal{S}$. Events may be simple or compound.
- Complement, intersection, and union of events.
- Two events are **Mutually Exclusive** or **Disjoint** if their intersection is empty.
- De Morgan's Laws.

Probability Axioms

Let $\mathcal{S}$ be the sample space and A_1, A_2 events.

1. $P(A) \geq 0$
2. $P(\mathcal{S}) = 1$
3. If A_1 and A_2 are disjoint, then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

Basic Properties

We can derive some basic properties from the three axioms previously described.

- $P(A') = 1 - P(A)$
- $0 \leq P(A) \leq 1$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

From De Morgan's laws:

- $P(A' \cap B') = P[(A \cup B)']$
- $P(A' \cup B') = P[(A \cap B)']$

Classical Definition of Probability

Laplace's definition:

The probability of an event is the ratio of the number of cases favorable to it, to the number of all cases possible when nothing leads us to expect that any one of these cases should occur more than any other, which renders them, for us, equally possible.

- $P(A) = \frac{\#A}{N}$, when each outcome is equally likely.
- Examples.

Counting Methods

The Fundamental Counting Principle

- If one event can occur in m ways and another independent event can occur in n ways, then the total number of outcomes for both events is $n \cdot m$.
- With more than two events, you just keep multiplying.
- Example: 3 types of buns (white, whole-grain, gluten free), 4 types of patties (beef, chicken, fish, veggie), and 2 sauces (ketchup, mustard). How many different burgers can you create? $3 \cdot 4 \cdot 2 = 24$.

Permutations

$$nPr = \frac{n!}{(n-r)!}$$

- There are 10 people in a room and three prizes. First prize is \$75, second prize is \$50, and third prize is \$25.
- In how many ways can we give out these prizes?

$$10P3 = \frac{10!}{(10-3)!} = 10 \cdot 9 \cdot 8 = 720$$

Combinations

$$nCr = \binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}$$

- There are 10 people in a room and three prizes. First prize is \$50, second prize is \$50, and third prize is \$50. Because all the prizes are the same, the distinction between first, second, and third does not matter. In this sense order does not matter.
- In how many ways can we give out three prizes of \$50?

$$\binom{10}{3} = \frac{10!}{(10-3)! \cdot 3!} = 120$$

Combinations II

- The number of combinations is lower than the number of permutations.
- Pascal's triangle and binomial coefficients.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Conditional Probability

Conditional Probability

For any two events A and B with $P(B) > 0$, the conditional probability of A given that B has occurred is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- Example.
- Diagrams.

Multiplication rule:

$$P(A \cap B) = P(B) \cdot P(A|B)$$

Law of Total Probability

Let A_1 and A_2 be two mutually exclusive and exhaustive events from the sample space $\mathcal{S}$. And let B be another event. Then:

$$P(B) = P(B|A_1) \cdot P(A_1) + P(B|A_2) \cdot P(A_2)$$

- Proof.
- More than two events.
- Intuition.

Let A_1 and A_2 be two mutually exclusive and exhaustive events from the sample space $\mathcal{S}$. And let B be another event. Then:

$$P(A_1|B) = \frac{P(B|A_1) \cdot P(A_1)}{P(B|A_1) \cdot P(A_1) + P(B|A_2) \cdot P(A_2)}$$

- Proof.
- Intuition.
- Prior probability and posterior probability.

Independence

- A and B are independent if $P(A|B) = P(A)$.
- This definition is equivalent to $P(A \cap B) = P(A) \cdot P(B)$.
- For independent events you may calculate the joint probability as the product of their marginal probabilities.
- More properties:
  - If A and B are independent, then A and B' are independent.
  - If A and B are independent, then A' and B' are independent.