Intertemporal Choice

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Two Periods

Problem

$$\max_{c_0, c_1} \quad u(c_0) + \beta u(c_1) \quad \text{s.t.} \quad c_0 + \frac{c_1}{1+r} = m$$

- Choose (c_0, c_1) to maximize lifetime utility subject to the budget constraint.
- $m \overset{\text{Def}}{=} m_0 + \frac{m_1}{1+r}$ is the present value of income.
- β the discount factor.
- r the real interest rate.

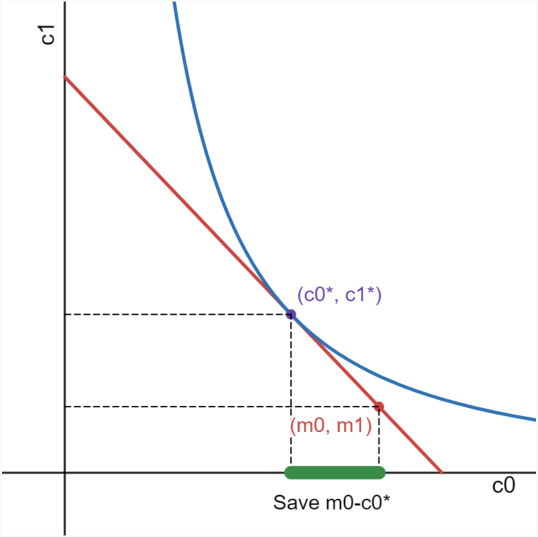
Euler Equation

$$\max_{c_0, c_1} \quad u(c_0) + \beta u(c_1) \quad \text{s.t.} \quad c_0 + \frac{c_1}{1+r} = m$$

- The first order conditions.

$$u'(c_0) = \beta(1+r) \cdot u'(c_1)$$
$$c_0 + \frac{c_1}{1+r} = m$$

Graphically - Case $m_0 >> m_1$



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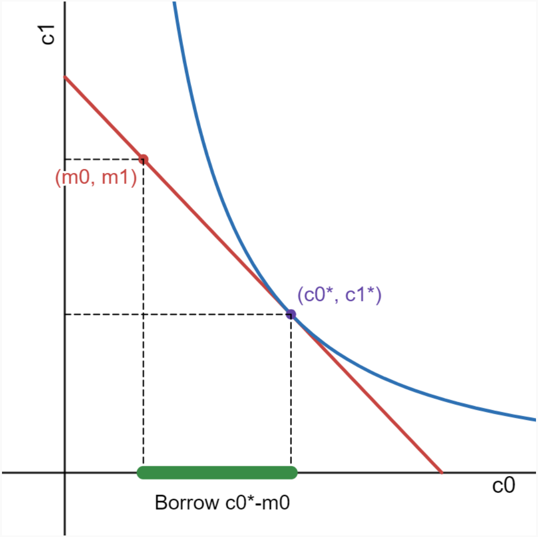

Timeline

	t=0	t=1
State:	z_0	z_1
Control:	c_0	c_1

- $z_0 \stackrel{\text{Def}}{=} m$
- Consume c_0 and save $z_0 - c_0$
- $z_1 \stackrel{\text{Def}}{=} (1 + r)(z_0 - c_0)$
- $c_1 = z_1$ and save nothing. Why?

Recursive Formulation

$$v_0(z_0) = \max_{c_0} \{ u(c_0) + \beta u(c_1) \}$$
$$\text{where} \quad c_1 = (1 + r)(z_0 - c_0)$$

- $v_0(z_0)$ is the value function of the problem.
- Give me your wealth in present value z_0 and I tell you the maximum utility you will get.

Envelope Result

$$\frac{dv_0(z_0)}{dz_0} = u'(c_0)$$

- Usual envelope result.
- Proof by differentiating $v_0(z_0)$ and using the first order condition stated before.
- An extra dollar in $t = 0$ increases my utility by $u'(c_0)$ which equals $\beta(1 + r) \cdot u'(c_1)$ because I redistribute that extra dollar optimally over time.

More Periods

Problem with 3 periods

$$\max_{c_0, c_1, c_2} \quad u(c_0) + \beta u(c_1) + \beta^2 u(c_2)$$
$$\text{s.t.} \quad c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} = m$$

- Let $m \stackrel{\text{Def}}{=} m_0 + \frac{m_1}{1+r} + \frac{m_2}{(1+r)^2}$ represent the present value of income over time.

First Order Conditions

$$
\begin{aligned}
u'(c_0) &= \beta(1+r)u'(c_1) \\
u'(c_1) &= \beta(1+r)u'(c_2) \\
c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} &= m
\end{aligned}
$$

Timeline

	t=0	t=1	t=2
State:	z_0	z_1	z_2
Control:	c_0	c_1	c_2

- $z_0 \stackrel{\text{Def}}{=} m$, consume c_0 and save $z_0 - c_0$.
- $z_1 \stackrel{\text{Def}}{=} (1+r)(z_0 - c_0)$, consume c_1 and save $z_1 - c_1$.
- $z_2 \stackrel{\text{Def}}{=} (1+r)(z_1 - c_1)$, consume $c_2 = z_2$ and save nothing.

Recursive Formulation

- We can break down this three period problem into two simpler problems as long as we go backwards.
- First consider periods 1 and 2, solve the problem and store the value function.
- Next consider periods 0 and 1.

Recursive Formulation - $t = 1$

$$v_1(z_1) = \max_{c_1} \left\{ u(c_1) + \beta u(c_2) \right\}$$
$$\text{where} \quad c_2 = (1 + r)(z_1 - c_1)$$

- Identical to the case $T = 2$.
- $v_1(z_1)$ measures the maximum utility I get from having z_1 dollars at that moment.
- Envelope result $v_1'(z_1) = u'(c_1)$.

Recursive Formulation - $t = 0$

$$v_0(z_0) = \max_{c_0} \left\{ u(c_0) + \beta v_1(z_1) \right\}$$
$$\text{where} \quad z_1 = (1+r)(z_0 - c_0)$$

- $v_1(\cdot)$ summarizes my optimal choice after $t = 1$.
- Choose between consuming more today, and having less z_1 for the future.
- Or consuming less today and having more z_1 for the future.

Recursive Formulation - $t = 0$

- Using $v_1'(z_1) = u'(c_1)$ the first order condition becomes:

$$u'(c_0) = \beta(1 + r) \cdot u'(c_1)$$

- The same Euler Equation as before.

Take Away

- Solve the full problem. This gives you two Euler Equations plus the intertemporal budget constraint.
- Solve two simplified problems using the recursive formulation. Each of them gives you an Euler Equation plus the intertemporal budget constraint.
- The recursive formulation is a powerful tool to solve more complex problems.

Problem with $T + 1$ periods

$$\max_{c_0, c_1, \ldots, c_T} \quad \sum_{t=0}^{T} \beta^t u(c_t) \quad \text{s.t.} \quad \sum_{t=0}^{T} \frac{c_t}{(1+r)^t} = z_0$$

- Start at $T - 1$ and solve the problem for $T - 1$ and T.
- Go backwards.
- You can break this down into $T - 1$ simple two-period problems.

Infinite Time

Utility and Budget Constraint

$$\sum_{t=0}^{\infty} \beta^t \cdot u(c_t)$$

- Need $\beta < 1$ for the sum to be well-defined.

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = m$$

- Assume the present value of income m is well defined.

Problem

$$\max_{\{c_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t \cdot u(c_t) \quad \text{s.t.} \quad \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = m$$

- Find the first order conditions.

$$u(c_t) = \beta(1+r) \cdot u(c_{t+1}) \quad \text{for} \quad t = 0, 1, 2, \ldots$$

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = m$$

Timeline

	t	t+1
State:	z_t	z_{t+1}
Control:	c_t	c_{t+1}

- z_t, the amount of money available at time t consume c_t and save $z_t - c_t$.
- $z_{t+1} \overset{\text{Def}}{=} (1 + r)(z_t - c_t)$, consume c_{t+1} and save $z_{t+1} - c_{t+1}$.
- ...

Recursive Formulation

$$v(z_t) = \max_{c_t} \left\{ u(c_t) + \beta v(z_{t+1}) \right\}$$
$$\text{where} \quad z_{t+1} = (1+r)(z_t - c_t)$$

- Envelope results still holds $v'(z_{t+1}) = u'(c_{t+1})$.
- Infinite time simplifies things: $v(\cdot)$ function in the RHS and LHS is the same function.
- Why?
- Bellman equation.

First Order Condition

$$u'(c_t) = \beta(1+r) \cdot u'(c_{t+1}) \quad \text{for} \quad t = 0, 1, 2, \ldots$$

- Same as before.
- We'll get one condition for every pair of periods.
- Combine with the budget constraint and find the optimal consumption over time.

Transversality Condition

$$\lim_{t \to \infty} z_t \geq 0$$

- Consumer do not carry debt at the end of her life.