Consumer Theory

Javier Tasso

May 29, 2024

University of Pennsylvania

Budget Constraint

Budget Constraint

$$p_1 x_1 + p_2 x_2 \leq m$$

- Consumption set $X = \mathbb{R}^2_+$.
- (x_1, x_2) are the quantities of good 1 and 2.
- (p_1, p_2, m) are their prices and income.
- Slope $-\frac{p_1}{p_2}$.
- X-intercept $\frac{m}{p_1}$.
- Y-intercept $\frac{m}{p_2}$.

Budget Constraint II

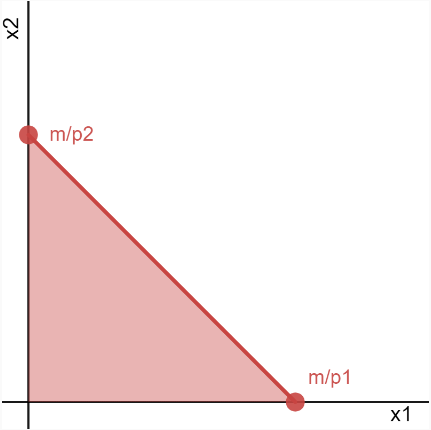

Figure 1: Budget Set

Budget Constraint III

- Changes in (p_1, p_2, m).
- The importance of relative prices.
- What does $\frac{dx_2}{dx_1} = -\frac{p_1}{p_2}$ mean?
    - The rate of change of the market.
- Tax consumption after a threshold.
- Rationing.
- Give some units for free.

Preferences

Weak Preference

- Consumption set $X = \mathbb{R}^2_+$.
- Consumer compares bundles $x = (x_1, x_2)$ and $y = (y_1, y_2)$ according to the weak preference relation $\succeq$.

$$x \succeq y \iff x \text{ is at least as good as } y$$

- $\succeq$ is the weak preference relation.
- Bundles are pretty general.

Strict Preference and Indifference

- If $x \succeq y$, but $y \not\succeq x$, then we say $x \succ y$. Bundle x is strictly preferred to bundle y.
- If $x \succeq y$ and $y \succeq x$, then we say $x \sim y$. Bundle x is indifferent to bundle y.

Rational Preferences

- Complete. Given any $x, y \in X$, we have $x \succeq y$ or $y \succeq x$ (or both).
    - Reflexive.
- Transitive. Given any $x, y, z \in X$ such that $x \succeq y$ and $y \succeq z$, then it must be that $x \succeq z$.

Typical Preferences

- Weak Monotonicity. Suppose $x >> y$, then $x \succ y$.
- Strong Monotonicity Suppose $x \geq y$ and $x \neq y$, then $x \succ y$.
- Convexity. Take $x, y, z \in X$ such that $y \succeq x$ and $z \succeq x$, then for $t \in [0, 1]$ we have $ty + (1 - t)z \succeq x$.
- Strict Convexity. Take $x, y, z \in X$ such that $y \succeq x$, $z \succeq x$, and $y \neq z$ then for $t \in (0, 1)$ we have $ty + (1 - t)z \succ x$.

Indifference Curves

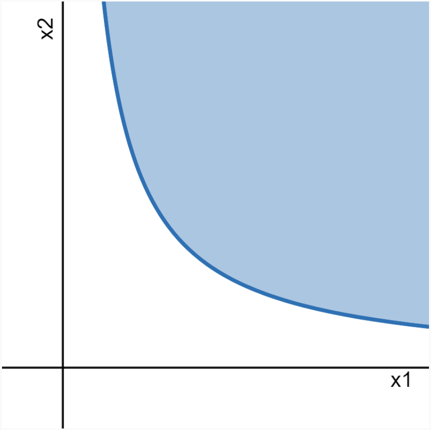

Figure 2: Indifference Curve

Indifference Curves II

- Because of transitivity:
    - Indifference curves do not cross each other.
- Because of monotonicity:
    - They are not fat.
    - They have a negative slope.
- Because of convexity:
    - They have a convex shape.
    - Its slope decreases (in absolute value) as we move right.

Marginal Rate of Substitution

$$\text{MRS} = -\frac{dx_2}{dx_1}$$

- The slope of the indifference curve.
- If I take one unit of good 1 from you, but I want you to remain indifferent... How many units of good 2 do I need to give you?
- The rate of change the consumer is willing to accept.

Utility

Utility Function

$$x \succeq y \iff u(x) \geq u(y)$$

- We assign a number to each bundle.
- Then we compare the numbers to determine if a bundle is preferred to another one.
- Ordinal: the actual number doesn't have a meaning.
    - A monotonic transformation of a utility function is another utility function that represents the same preferences.

Representation

- If preferences are rational and continuous, then there is a continuous utility function that represents them.
- Continuity of preferences is a technical assumption that guarantees we can represent a rational preference relation with an utility funciton.

Typical Preferences

Utility functions that represent typical preferences.

- Cobb-Douglas: $u(x_1, x_2) = x_1 x_2$
- Perfect Complements: $u(x_1, x_2) = \min\{x_1, x_2\}$
- Perfect Substitutes: $u(x_1, x_2) = x_1 + x_2$
- Quasi-Linear: $u(x_1, x_2) = v(x_1) + x_2$
    - $u(x_1, x_2) = \ln(x_1) + x_2$

Marginal Utility

$$MU_{x_1} = \frac{\partial u(x_1, x_2)}{\partial x_1}$$

- When this is possible!
- How much extra utility I get by consuming more x_1, keeping the consumption of x_2 constant.
- Usually positive. Why?
- Usually decreasing. Why?
- Its value will scale up/down with transformations to $u(x_1, x_2)$.

Marginal Utility and Marginal Rate of Substitution

$$\text{MRS} = -\frac{dx_2}{dx_1} = \frac{MU_{x_1}}{MU_{x2}}$$

- Get the MRS by dividing the marginal utilities.
- Because we are dividing, the MRS will **not** scale up/down with transformations to $u(x_1, x_2)$.
- Why? Proof differencing the utility function.

Utility Maximization Problem

Statement of the problem

$$\max_{x_1, x_2} \quad u(x_1, x_2) \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 \leq m$$

- Because of monotonicity, the consumer will spend all her money.
- Its solution is the individual demand for each of the goods.

Graphically

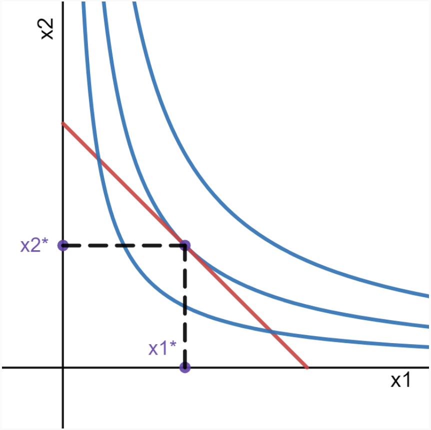

Figure 3: Utility Maximization Problem

Analytically

- We call its solution $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$ the Marshallian or ordinary demand.
- The maximum utility attained $v(p_1, p_2, m)$ is called indirect utility function.
- These objects are well-defined because we are maximizing a continuous function over a compact set.

Characterization

$$\max_{x_1, x_2, \lambda} \quad L = u(x_1, x_2) + \lambda(m - p_1 x_1 - p_2 x_2)$$

Let λ be the Lagrange multiplier, then the first order conditions:

$$\begin{aligned} \text{MU}_1 &= \lambda p_1 \\ \text{MU}_2 &= \lambda p_2 \\ p_1 x_1 + p_2 x_2 &= m \end{aligned}$$

$(x_1(p_1, p_2, m), x_2(p_1, p_2, m), \lambda^*)$ is the solution.

Characterization II

- If the MRS is defined and the solution is interior, then:

$$\text{MRS} = \frac{p_1}{p_2}$$

- The rate at which the market trades goods 1 and 2 must equal the rate at which the consumer is willing to make that trade?
- Why? Assume not.

Characterization III

- If the utility function is differentiable and the solution is interior, then:

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2}$$

- Marginal utility per dollar must be equal for all goods.

Shadow Price

$$v(p_1, p_2, m) = u(x_1(p_1, p_2, m), x_2(p_1, p_2, m))$$
$$E(p_1, p_2, m) = p_1 x_1(p_1, p_2, m) + p_2 x_2(p_1, p_2, m)$$

- $\frac{\partial v(\cdot)}{\partial m} = \lambda^* p_1 \cdot \frac{\partial x_1(\cdot)}{\partial m} + \lambda^* p_2 \cdot \frac{\partial x_2(\cdot)}{\partial m}$
- $\frac{\partial E(\cdot)}{\partial m} = p_1 \cdot \frac{\partial x_1(\cdot)}{\partial m} + p_2 \cdot \frac{\partial x_2(\cdot)}{\partial m}$

Now construct the ratio $\frac{\partial v(\cdot)}{\partial m} / \frac{\partial E(\cdot)}{\partial m} = \lambda^*$. An extra unit of optimal expenditure leads to λ^* units of extra utility.

Properties of the Marshallian demand

- Continuous.
- Usually decreasing in its own price, but not always.
    - Typical/Giffen goods.
- Can increase/decrease with income.
    - Normal/Inferior goods.
- Homogeneous of degree 0 in (p_1, p_2, m).
    - There is no money illusion.
- Walras' law.

Properties of the Indirect Utility function

- Continuous.
- Increases with m.
- (weakly) Decreases with p_1 and p_2.
- Homogeneous of degree 0 in (p_1, p_2, m).
- Quasi-convex.
- Roy's identity.

$$x_1(p_1, p_2, m) = -\frac{\frac{\partial v(p_1,p_2,m)}{\partial p_1}}{\frac{\partial v(p_1,p_2,m)}{\partial m}}$$

UMP for typical preferences

- Cobb-Douglas.
- Perfect Complements.
- Perfect Substitutes.
- Quasi-Linear.

Expenditure Minimization Problem

Introduction

- An increase in p_1 affects $x_1(p_1, p_2, m)$ in two ways:
    1. By making good 1 more expensive.
    2. By decreasing the consumer's real income.
- So I can't tell for sure what's driving the change in demand of good 1. Is it p_1 or is it the fact that real income changed?
- The second effect is called income effect.
- Goal: Isolate the effect of a price change.

Statement of the Problem

$$\min_{x_1,x_2} \quad p_1x_1 + p_2x_2 \quad \text{s.t.} \quad u(x_1, x_2) \geq u$$

- No excess utility (continuity of $u(\cdot)$).
- Target utility level u.
- Consumer wants to achieve u by spending at little as possible.
- The solution gives me the quantities of goods 1 and 2 to achieve u spending as little as possible.

Graphically

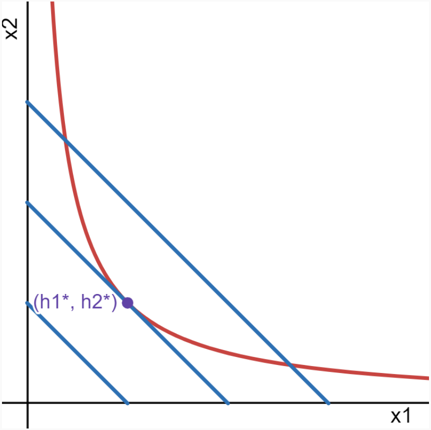

Figure 4: Expenditure Minimization Problem

Analytically

$$\min_{x_1,x_2} \quad p_1x_1 + p_2x_2 \quad \text{s.t.} \quad u(x_1, x_2) \geq u$$

- We call its solution $h_1(p_1, p_2, u)$ and $h_2(p_1, p_2, u)$ the Hicksian or compensated demand.
- The minimum expenditure $e(p_1, p_2, u)$ achieved is the expenditure function.

Characterization

$$\min_{x_1,x_2,\lambda} \quad L = p_1x_1 + p_2x_2 + \lambda(u - u(x_1,x_2))$$

Where λ is the Lagrange multiplier. The FOC:

$$\begin{aligned} p_1 &= \lambda \mathrm{MU}_1 \\ p_2 &= \lambda \mathrm{MU}_2 \\ u(x_1,x_2) &= u \end{aligned}$$

$(h_1(p_1,p_2,u), h_2(p_1,p_2,u), \lambda^*)$ is the solution.

Characterization II

- If the MRS is defined and the solution is interior, then:

$$\text{MRS} = \frac{p_1}{p_2}$$

- Same condition we've seen before.

Properties of the Hicksian Demand

- Homogeneous of degree 0 in p.
- Compensated law of demand.
    - We cannot know for sure the relationship between p_1 and $x_1(p_1, p_2, m)$.
    - We actually can make statements about the relation between p_1 and $h_1(p_1, p_2, u)$.
    - Very important property.

Compensated Law of Demand

$$(p_1' - p_1) \cdot [h_1(p_1', p_2, u) - h_1(p_1, p_2, u)] \leq 0$$

- Proof.
- Corollary: Own price effects are (weakly) negative.
- Hicksian demands have a negative slope.
- Marshallian demand may not have a negative slope (Giffen goods) because of the income effect.
- Plots.

Properties of the Expenditure Function

- Continuous.
- Homogeneous of degree 1 in p.
- Increasing in u.
- (weakly) increasing in p.
- Concave in p.
- Shephard's lemma:

$$h_1(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_1}$$

EMP for typical preferences

- Cobb-Douglas.
- Perfect Complements.
- Perfect Substitutes.
- Quasi-Linear.

Duality & Slutsky Equation

Duality

- If I solve the EMP at the utility level I would get if I had income m, the solution is exactly m.

$$e[p_1, p_2, v(p_1, p_2, m)] \equiv m$$

- If I solve the UMP at the expenditure associated with utility level u, the solution is exactly u.

$$v[p_1, p_2, e(p_1, p_2, u)] \equiv u$$

- In words $e(\cdot, u)$ and $v(\cdot, m)$ are inverses of each other.

Slutsky Equation

$$\frac{\partial h_1(p,u)}{\partial p_i} = \frac{\partial x_1(p,m)}{\partial p_i} + \frac{\partial x_1(p,m)}{\partial m} \cdot x_i(p,m)$$

- Here $i = 1, 2$, any of them.
- Similar equation for good 2.
- Proof using duality and Shephard's lemma.

Slutsky Equation II

$$\frac{\partial x_1(p,m)}{\partial p_i} = \frac{\partial h_1(p,u)}{\partial p_i} - \frac{\partial x_1(p,m)}{\partial m} \cdot x_i(p,m)$$

- RHS, total effect. Include both prices and income.
- LHS, first term. Substitution effect. Isolates the effect of a price change.
- LHS, second term. Income effect.

$$\text{TE} = \text{SE} \pm \text{IE}$$

Slutsky Equation III

$$\frac{\partial x_1(p,m)}{\partial p_1} = \frac{\partial h_1(p,u)}{\partial p_1} - \frac{\partial x_1(p,m)}{\partial m} \cdot x_1(p,m)$$

- Own price.
- Compensated law of demand implies SE is negative.
- If good 1 is a normal good, then $x_1(p,m)$ has a negative slope.

Law of Demand

- Normal good $\implies$ Typical good
- Giffen good $\implies$ Inferior good.
- Three graphs.
- Ordinary demand can have positive/negative slope. If we know it is a normal good, then it has a negative slope for sure.

Welfare Analysis

Changes

- In general. From some status quo to a new situation.

$$(p_1^0, p_2^0, m^0) \to (p_1', p_2', m')$$

- Usually we focus on price changes:

$$(p_1^0, p_2^0, m) \to (p_1', p_2', m)$$

- Notation:

$$p^0 \to p'$$

- To fix ideas, we shall assume only p_1 changes. If it's useful we normalize $p_2^0 = p_2' = 1$.

Equivalent Variation

- Suppose prices change from p^0 to p'.
- At prices p^0, the consumer gets utility u^0. At prices p', the consumer gets u'.

$$v(p^0, m + \text{EV}) = u'$$

- In words. The price change has not happened yet. The EV is the extra money I need to give to the consumer so he's willing to accept the change.

Equivalent Variation II

$$v(p^0, m + \text{EV}) = u'$$

- Suppose from p^0 to p' prices are increasing (decreasing). The change has not happened yet.
- The consumer would be worse off (better off) with the change.
- To make him willing to accept the change, I need to give him money (take money from him).
- How much money? The EV.

Equivalent Variation III

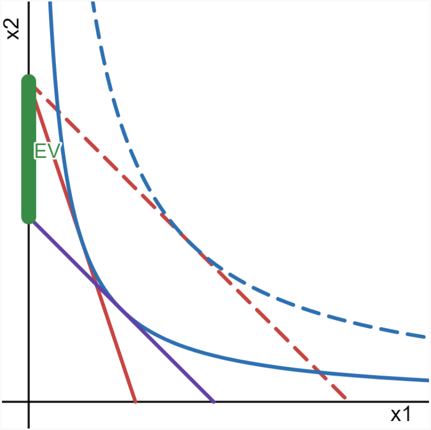

Figure 5: Equivalent Variation. Assume $p_2 = 1$.

Equivalent Variation IV

$$\text{EV} = e(p^0, u') - m$$

- Positive if consumer is better off with the change. Negative otherwise.
- Note that we always have p^0. So everything is using the original prices.

Equivalent Variation V

- Suppose only p_1 changes.
- p_2 and u' are fixed.
- The only thing that is changing is p_1.

$$
\begin{aligned}
\text{EV} &= e(p_1^0, p_2^0, u') - m \\
&= e(p_1^0, p_2^0, u') - e(p_1', p_2^0, u') \\
&= \int_{p_1'}^{p_1^0} h_1(t, p_2, u')dt
\end{aligned}
$$

Equivalent Variation VI

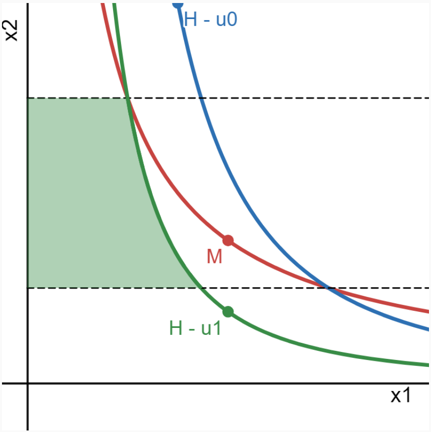

Figure 6: Equivalent Variation

Compensating Variation

- Suppose prices change from p^0 to p'.
- At prices p^0, the consumer gets utility u^0. At prices p', the consumer gets u'.

$$v(p', m - \text{CV}) = u^0$$

- In words. After the price change, the CV is the amount of money I need to give you (take from you) so you reach the original utility you had before the change.

Compensating Variation II

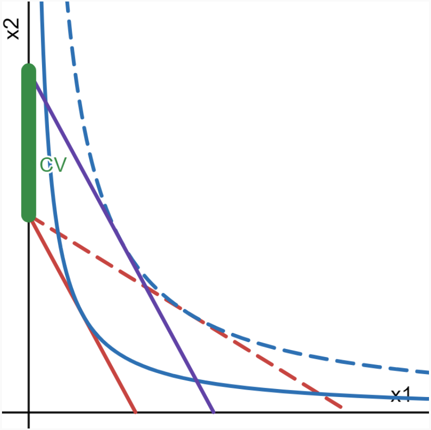

Figure 7: Compensating Variation. Assume $p_2 = 1$

Compensating Variation III

$$\text{CV} = m - e(p', u^0)$$

- If the price change is good for the consumer, $\text{CV} > 0$.
- If it is bad, $\text{CV} < 0$.
- We always have p'. Everything is using the new prices.

Compensating Variation V

- Suppose only p_1 changes.
- p_2 and u^0 are fixed.
- The only thing that is changing is p_1.

$$
\begin{aligned}
\mathrm{CV} &= m - e(p_1', p_2^0, u^0) \\
&= e(p_1^0, p_2^0, u^0) - e(p_1', p_2^0, u^0) \\
&= \int_{p_1'}^{p_1^0} h_1(t, p_2, u^0) dt
\end{aligned}
$$

Compensating Variation VI

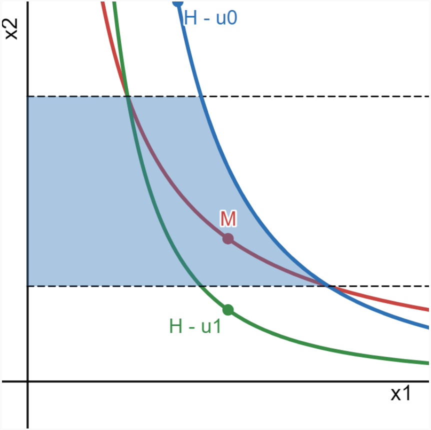

Figure 8: Compensating Variation

Change in the CS

- Not an exact measure of welfare.
- Can be used to approximate EV and CV when income effect is low.
- Lies between the EV and CV.
    - For normal goods: $|CV| \geq |\Delta CS| \geq |EV|$
        - Why? CV is large because after a price increase there are two effects: (*i*) the price effect itself and (*ii*) the income effect. If I want to compensate the consumer, I need to give him extra extra money.
    - For inferior goods: $|EV| \geq |\Delta CS| \geq |CV|$
        - Why? CV is small...

Change in the CS II

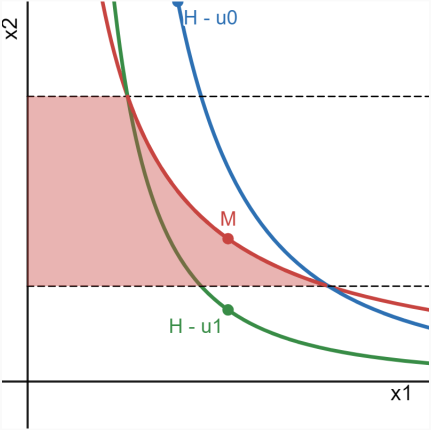

Figure 9: Change in the CS

Change in the CS III

- If there is no income effect, then $\text{EV} = \text{CV} = \Delta\text{CS}$.
- Why? Because with no income effect marshallian demand and hicksian demand are the same.
- Example: quasi-linear preferences in the interior solution.