Microeconomics
Homework 6: Producer Theory

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1. Consider the Cobb-Douglas production function $f(k,l) = Ak^{\alpha}l^{\beta}$ for constants $A, \alpha, \beta > 0$.
   (a) For what values of $\alpha + \beta$ does this production function has constant, decreasing and increasing returns to scale?
   (b) Set $\alpha = \beta = 0.5$. Plot the marginal product of labor $\text{MP}_{\text{L}} = \frac{\partial f(k,l)}{\partial l}$ when $k = 1$ and $A = 1$ or $A = 2$.
   (c) The parameter A is called total factor productivity (TFP). Intuitively explain how A affects the marginal product of labor and the marginal product of capital.
2. Consider the production function $f(k,l) = \min\{k,l\}$.
   (a) Argue this production function has constant returns to scale.
   (b) Fix $k = 4$ and plot the marginal product of labor.
3. Consider the production function $f(k,l) = \big(\min\{k,l\}\big)^{\gamma}$ with $\gamma > 0$.
   (a) Classify the returns to scale according to the value of γ.
   (b) Solve the cost minimization problem.
   (c) Find the marginal cost.
   (d) Fix $w = r = 1/2$. Plot the marginal cost when $\gamma = 1/2$, $\gamma = 1$, and $\gamma = 2$.
   (e) Set $w = r = 1/2$, $\gamma = 1/2$, and $p = 4$, what's the quantity that maximizes profits?
   (f) Explain why the profit maximization problem won't have a solution if $\gamma > 1$.
   (g) Assume $\gamma < 1$ and solve the profit maximization problem.
4. Given the production function $f(k,l) = \sqrt{k} + \sqrt{l}$.
   (a) Solve the profit maximization problem: $\max_{k,l} pf(k,l) - wl - rk$.
   (b) Find the supply of the firm $y(p,w,r)$.
   (c) Set $w = r = 1$ and $p = 10$. How many units does this firm produce? Find the total cost and the profits.
   (d) With l in the horizontal axis and k in the vertical plot the isoquant and isocost lines. Verify that the firm is effectively minimizing costs.
   (e) What's the value of the marginal rate of technical substitution at the optimal solution?
5. The cost function is $c(q) = 0.1q^2 + 10q + 50$. Suppose the firm takes p as given. Find the supply curve.
6. Suppose $y = f(k,l)$ is constant returns to scale. Denote f'_K and f'_L the partial derivatives. Show that $y = f'_K \cdot k + f'_L \cdot l$.

7. Suppose $y = f(k, l) = \sqrt{kl}$.
   (a) Determine the returns to scale.
   (b) Solve the cost minimization problem.
   (c) Whenever possible, find the supply of the firm.

8. Suppose $y = f(k, l) = (kl)^{1/4}$.
   (a) Determine the returns to scale.
   (b) Solve the cost minimization problem.
   (c) Verify Shephard's lemma.
   (d) Solve the profit maximization problem. Find the supply of the firm and the profit function.
   (e) Verify Hotteling's lemma.

9. For the following production functions solve the cost minimization problem. The parameter $\gamma > 0$ controls the returns to scale.
   - Cobb Douglas: $f(k, l) = k^{\alpha} l^{\beta}$ with $\alpha, \beta > 0$.
   - Perfect Substitutes: $f(k, l) = (k + l)^{\gamma}$
   - Perfect Complements: $f(k, l) = (\min\{k, l\})^{\gamma}$

Answers

1. (a) $\alpha + \beta = 1$ means constant returns to scale. $\alpha + \beta < 1$ is decreasing returns to scale.
   (b) See figure.
   (c) $\mathrm{MP_L} = A\beta k^{\alpha} l^{\beta-1}$ and similarly for hte marginal product of capital. An increase of the TFP increases both marginal products of labor and capital.

2. (a) Multiply both K and L by some constant $\lambda > 0$. You can pull out the constant from the $\min\{\cdot,\cdot\}$.
   (b) See graph.

3. (a) $\gamma = 1$ is CRS. $\gamma > 1$ is IRS.
   (b) $c(q, w, r) = (w + r)q^{1/\gamma}$, with $K(w, r, q) = L(w, r, q) = q^{1/\gamma}$
   (c) $mc(q, w, r) = (w + r)\frac{1}{\gamma} q^{\frac{1}{\gamma}-1}$
   (d) See graph.
   (e) $q^* = 2$.
   (f) If $\gamma > 1$ there are increasing returns to scale and the marginal cost is decreasing with output. A competitive firm that takes the price given would like to produce as much as possible.
   (g) $l(p, w, r) = k(p, w, r) = \left(\frac{\gamma p}{w+r}\right)^{\frac{1}{1-\gamma}}$ the supply is $y(p, w, r) = \left(\frac{\gamma p}{w+r}\right)^{\frac{\gamma}{1-\gamma}}$

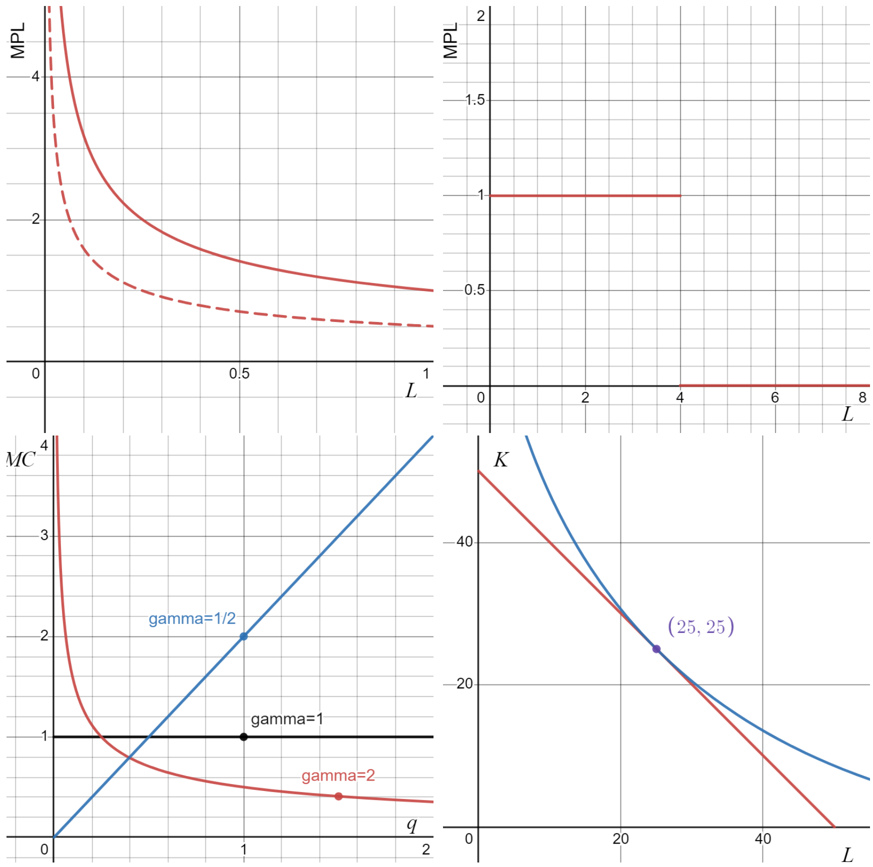

Figure 1: Exercises 1, 2, 3, and 4

4. (a) $K(p, w, r) = \left(\frac{p}{2r}\right)^2$, $L(p, w, r) = \left(\frac{p}{2w}\right)^2$
   (b) $y(p, w, r) = \frac{p}{2r} + \frac{p}{2w}$
   (c) $y(10, 1, 1) = 10$ using $K(10, 1, 1) = 25$ and $L(10, 1, 1) = 25$. The total cost is 50 and the profits are $100 - 50 = 50$.

(d) See graph.

(e) MRTS $= \frac{\text{MP}_L}{\text{MP}_K} = \left(\frac{K}{L}\right)^{\frac{1}{2}}$. In the optimal values this is just 1 and it matches the price ratio w/r.

5. $y(p) = \max\{5(p-10), 0\}$.

6. Start with $F(\lambda K, \lambda L) = \lambda F(K, L)$. Differentiate both sides with respect to λ. Evaluate at $\lambda = 1$.

$$F(\lambda K, \lambda L) = \lambda F(K, L)$$
$$\frac{\partial F(\lambda K, \lambda L)}{\partial K} \cdot K + \frac{\partial F(\lambda K, \lambda L)}{\partial L} \cdot L = F(K, L)$$
$$\frac{\partial F(K, L)}{\partial K} \cdot K + \frac{\partial F(K, L)}{\partial L} \cdot L = F(K, L)$$

7. (a) Constant.
   (b) $K(\cdot) = \sqrt{\frac{w}{r}} \cdot q$, $L(\cdot) = \sqrt{\frac{r}{w}} \cdot q$, and $c(q, w, r) = 2 \cdot \sqrt{wr} \cdot q$
   (c) Set $p = mc$ whenever possible. Note that the marginal cost is $2\sqrt{wr}$.

$$y(p) = \begin{cases} 0 & \text{if} \quad p < 2\sqrt{wr} \\ [0, \infty) & \text{if} \quad p = 2\sqrt{wr} \\ \text{undefined} & \text{if} \quad p > 2\sqrt{wr} \end{cases}$$

8. (a) Decreasing.
   (b) $L(\cdot) = \sqrt{\frac{r}{w}} \cdot q^2$, $K(\cdot) = \sqrt{\frac{w}{r}} \cdot q^2$, and $c(q, r, w) = 2\sqrt{wr}q^2$
   (c) Just take derivatives of the cost function with respect to w and r.
   (d) $y(p, w, r) = \frac{p}{4\sqrt{rw}}$ and $\pi(p, w, r) = \frac{p^2}{8\sqrt{rw}}$.
   (e) Just take derivatives of the profit function with respect to p.

9. You can apply Shephard's lemma to find the conditional input demands.

- $l^C(w, r, q) = \left(\frac{r\beta}{w\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \cdot q^{\frac{1}{\alpha+\beta}}$ and $k^C(w, r, q) = \left(\frac{w\alpha}{r\beta}\right)^{\frac{\beta}{\alpha+\beta}} \cdot q^{\frac{1}{\alpha+\beta}}$
- $c(q, w, r) = \min\{w, r\} \cdot q^{\frac{1}{\gamma}}$
- $c(q, w, r) = (w + r) \cdot q^{\frac{1}{\gamma}}$